KV-THEORY OF CATEGORIES

BY

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ABSTRACT. Quillen has constructed a K-theory K_*C for nice categories, one of which is the category of projective R-modules. We construct a theory KV_*C for the nice categories parametrized by rings. When applied to projective modules we recover the Karoubi-Villamayor K-theory $KV_*(R)$.

As an application, we show that the Cartan map from $K_*(R)$ to $G_*(R)$ factors through the groups $KV_*(R)$. We also compute KV_* for the categories of faithful projectives and Azumaya algebras, generalizing results of Bass.

The purpose of this paper is to provide enough tools to make the Karoubi-Villamayor groups $KV_*(R)$ as manipulable as Quillen's groups $K_*(R)$. Since the KV_* are much closer to a cohomology theory than the K_*, it should be easier to compute the values of the groups $KV_*(R)$. The Gersten-Anderson spectral sequence relating these groups should then allow us to detect nontrivial elements in $K_*(R)$, and to look for new structural results.

Quillen has defined a K-theory for two kinds of "nice" categories: exact categories and symmetric monoidal categories (with faithful translations). Now a fundamental property of Karoubi-Villamayor functors should be that (i) they depend on a ring R, and (ii) they do not change if R is replaced by $R[t]$. In order to accomodate this property, we need to consider categories parametrized by rings. Fortunately, there is an abundance of such categories. In [Q] we encounter the exact categories M, P, H_S and M_p; in [B1] we encounter the symmetric monoidal categories Iso(P), FP, Pic, Az, Quad, etc.

After some preliminaries, the definitions are given in §2. The groups KV_* are the homotopy groups of an infinite loop space, bisimplicially defined. We show that they possess all the expected properties: cofinality, homotopy invariance, long exact sequences, and an analogue of the Gersten-Anderson spectral sequence. With care, the tools of characteristic filtration, resolution, devissage, localization, etc. all go through in this setting.

One unexpected benefit of this approach is that the Cartan map $K_*(R) \to G_*(R)$ factors through the groups $KV_*(R)$ for all Noetherian rings R. This is proven in §3.

We conclude by computing the groups KV_*S for many of the symmetric monoidal categories mentioned above. The idea is that in these cases we can construct an analogue of Gl, providing a simplicial group delooping of the relevant

infinite loop spaces. These computations were originally made in the author's dissertation, in a more primitive form.

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1. Preliminaries. In this section we collect several folklore-type results on K-theory and symmetric monoidal categories that we shall need.

Any small category C gives rise to a topological space BC by taking the geometric realization of the nerve of the category. This is a CW complex whose vertices and edges "are" the objects and morphisms of S.

Recall that a *symmetric monoidal category* is a category S with a unit 0:(terminal category) $\to S$ and a product $\square$: $S \times S \to S$, together with natural transformations $S \square 0 \cong S \cong 0 \square S$, $S \square (S \square S) \cong (S \square S) \square S$, and $S \square S \cong S \square S$ (with factors switched). There are certain coherence diagrams which are required to commute; these may be found in [**Mac**]. If, in addition, the product is associative and 0 is a strict two-sided unit, we call S a *permutative category*.

An important example of a symmetric monoidal category is the category of finite based sets and based maps, under the wedge product. For historical reasons this category is denoted Γ^{op} (cf. [**S**]). The skeletal subcategory on $\mathbf{0} = \{*\}$, $\mathbf{1} = \{*, 1\}$, etc. is a permutative category if we choose a lexicographical wedge product. The simplicial category Δ^{op} is the subcategory on objects $\mathbf{n}$ and maps which increase up to some point, and are zero thereafter. More information can be found in [**A1**] and [**S**].

A Γ-*space* (see [**S**]) is a functor $F: \Gamma^{op} \to Top$ for which $F(\mathbf{0})$ is contractible and the natural maps $F(X \vee Y) \to F(X) \times F(Y)$ are homotopy equivalences. A Γ-*category* is a functor from Γ^{op} to *Cat* whose composition with realization is a Γ-space.

It is immediate from the definitions that the data describing a symmetric monoidal category S suffice to make BS a homotopy associative, homotopy commutative H-space. In fact, the coherence conditions make it suitable food for infinite loop machines:

THEOREM 1.1. *If S is a small symmetric monoidal category, BS is an E_∞-space. There is a Γ-category C with $BC(\mathbf{1}) \simeq BS$.*

This was proven in [**M3**], [**T1**], and [**T2**, p. 106]. The general idea of the Γ-space structure is this. There is a function $\Psi: \Gamma^{op} \to Cat$ given by $\Psi(\mathbf{1} \vee \cdots \vee \mathbf{1}) = S \times \cdots \times S$. If α is a map in Γ^{op}, $\Psi(\alpha)$ is given by an incidence matrix. Ψ fails to be a Γ-category because there is only a natural equivalence between $\Psi(\alpha\beta)$ and $\Psi(\alpha)\Psi(\beta)$ instead of equality (Ψ is an "op-lax" functor). Ψ does retain enough information to give $B\Psi$ a Γ-space structure up to homotopy equivalence. Thomason "rectifies" Ψ to obtain a Γ-category by replacing S by a more structured (but homotopy equivalent) category.

COROLLARY 1.2. *If S is a symmetric monoidal category, BS is an infinite loop space if and only if $\pi_0(BS)$ is a group.*

PROOF. Consider a CW complex H which is a homotopy-associative H-space. Then H possesses a homotopy inverse just in case $\pi_0(H)$ is a group. This is an observation of G. Whitehead, and may be seen by combining III(4.17) and X(2.2) of [**Wh**]. On the other hand, Segal has shown in [**S**] that if F is a Γ-space and $H = F(1)$ has a homotopy inverse, then H has an infinite loop space structure. These remarks, applied to $F = BC$ and $H = BC(1)$, prove the corollary.

Here are two applications to K-theory. Let M be a small exact category in the sense of [**Q**]. The "Quillen construction" yields a new category QM, which is symmetric monoidal with product $Q(\oplus)$. Since it is connected, BQM is an infinite loop space.

Quillen has given an entirely different construction for K-theory in [**GQ**]. Let *SMCat* denote the category of small symmetric monoidal categories S which satisfy
(i) all maps are isomorphisms,
(ii) all translations $\square b\colon \mathrm{Aut}_S(a) \to \mathrm{Aut}_S(a\square b)$ are injections.
Given such an S, Quillen has defined a new symmetric monoidal category S^+ (denoted $S^{-1}S$ in [**GQ**]). Note that S^+ does not belong to *SMCat*. Quillen proves that $\pi_0(BS^+)$ is the Grothendieck group $K_0(S)$. It follows that BS^+ is an infinite loop space.

Quillen also proves that $B(S^+)$ is the "group completion" of BS in the sense of [**M3**]. That is, the map $S \to S^+$ induces an isomorphism of $H_*(BS^+)$ with

$$H_*(BS)\left[\pi_0^{-1}S\right] = \mathrm{colim}\; \amalg H_*(B\mathrm{Aut}_S(s)),$$

where the colimit runs over the objects s of the translation category of $\pi_0(S)$. If we let B_0S^+ denote the basepoint component of BS^+, then it follows that $H_*(B_0S^+) = \mathrm{colim}\; H_* B\mathrm{Aut}_S(s)$.

PROPOSITION 1.3. (i) *Quillen's $K_i(S)$ agrees with the $K_i(S)$ defined by Bass (in* [**B2**, *Chapter* VIII]*) for $i = 0, 1$.*

(ii) ("*Cofinality*") *Let $S \to T$ be a full and faithful cofinal functor in SMCat. Then $B_0S^+ \cong B_0T^+$, and $K_*(S) \cong K_*(T)$ for all $* > 1$.*

PROOF. We have already observed that $\pi_0(BS^+)$ is the Grothendieck group $K_0(S)$ of Bass. We also have

$$K_1(S) = \pi_1(B_0S^+) = H_1(B_0S^+) = \mathrm{colim}\; H_1(B\mathrm{Aut}(s))$$
$$= \mathrm{colim}(\mathrm{Aut}(s)/[\mathrm{Aut}(s), \mathrm{Aut}(s)]),$$

which is Bass' definition of $K_1(S)$.

Now suppose $f\colon S \to T$ is a cofinal functor, i.e., that for every $t \in T$ some $t\square t'$ is an $f(s)$. Then $H_*(B_0T^+) = \mathrm{colim}\; H_* B\mathrm{Aut}_T(f(s))$, the colimit being taken over $\pi_0(S)$. If f is full and faithful, then $\mathrm{Aut}_S(s) = \mathrm{Aut}_T(f(s))$ and, consequently, $H_*(B_0S^+) \to H_*(B_0T^+)$ is an isomorphism. As $B_0S^+ \to B_0T^+$ is an H-space map, it is a homotopy equivalence, so that for $* > 1$ we have $K_*(S) = \pi_*(B_0S^+) = \pi_*(B_0T^+) = K_*(T)$.

REMARK. Bass has also defined $K_2(S)$ in [**B3**] to be the colimit, again over the translation category of $\pi_0(S)$, of the groups $H_0(\text{Aut}(s); [\text{Aut}(s), \text{Aut}(s)])$. It is proven in [**We2**] that Bass' $K_2(S)$ agrees with Quillen's $K_2(S)$.

The glue between the Q and S^+ constructions is provided by the following observation. Suppose that P is an exact category in which all of the distinguished short exact sequences split. By ignoring all but the isomorphisms of P we can form the symmetric monoidal category $\text{Iso}(P)$. The spaces ΩBQP and $B(\text{Iso}(P)^+)$ are homotopy equivalent (see [**GQ**]).

If the category S is suitably nice, we can construct a group $\text{Aut}(S)$ playing the role of $\text{Gl}(R)$, and prove an analogue of the $B\text{Gl}(R)^+ = B_0(S^+)$ theorem, where S is the category $\amalg \text{Gl}_n(R)$. For convenience, we will work in the category $SPCat$ of all small, skeletal symmetric monoidal categories in $SMCat$ for which
  (i) "$\Box 0$" is the identity on $\text{Aut}(s)$,
  (ii) the translations "$\Box(t\Box u)$" and "$(\Box t)\Box u$" from $\text{Aut}(s)$ to $\text{Aut}(s\Box t\Box u)$ agree.

The point is that $\text{Aut}_S(-)$ is a group-valued functor on the (directed) translation category of the monoid $\pi_0 S$. We define $\text{Aut}(S)$ to be the colimit of this functor. Note that $\text{Aut}(\amalg \text{Gl}_n)$ recovers the group Gl. Typical members of $SPCat$ are the skeletal permutative categories in $SMCat$.

We need to describe the "group completion" of a connected space X. Let us define the group completion of X to be a homology isomorphism $X \to Y$, where Y is an H-space. The prototype is the group completion $B\text{Gl}(R) \to B\text{Gl}^+(R) \simeq \Omega_0 BQ\mathbf{P}(R) \simeq B_0\text{Iso }\mathbf{P}(R)^+$, where $\mathbf{P}(R)$ is the exact category of finitely generated projective R-modules.

Not every connected space has a group completion. If the group completion exists, though, it is unique and characterized by the following property:

LEMMA 1.4. *If it exists, the group completion $X \to Y$ is universal for maps from X into H-spaces, and is unique up to homotopy equivalence. That is, for all H-spaces H the map $[Y, H] \to [X, H]$ is a group isomorphism.*

PROOF. The cofibre Y/X is acyclic and simply connected (e.g., by van Kampen's theorem), so Y/X is contractible. The result now follows from the exact cofibration sequence of groups: $[Y/X, H] \to [Y, H] \to [X, H] \to [Y/X, \Omega H]$.

We can now state the "$B\text{Aut}(S)^+ = B_0(S^+)$" result:

PROPOSITION 1.5. *If $S \in SPCat$, then $B_0 S^+$ is the group completion of $B\text{Aut}(S)$. Moreover,*

$$K_1(S) = \text{Aut}(S)/[\text{Aut}(S), \text{Aut}(S)].$$

PROOF. Copying the telescope construction of [**GQ**, p. 224] produces an acyclic map $B\text{Aut}(S) \to B_0 S^+$. This map is realized by defining a functor from the translation category of $\pi_0(S)$ to Cat (send s to $\text{Aut}(s)$), constructing the corresponding cofibered category L (q.v. [**T2**, p. 92]), and defining $L \to S^+$ by $(s, s') \mapsto (s, s')$. The composite $B\text{Aut}(S) \simeq BL \to B_0 S^+$ is acyclic for the reasons cited in [**GQ**]. Since $B_0 S^+$ is an H-space, the map must be a group completion. Reading the H_1 isomorphism gives the last equation.

REMARK 1.6. There is another case, given by Bass in [B2, p. 355], in which we can define $\text{Aut}(S)$. This case includes all S in *SMCat* with a countable number of objects. This $\text{Aut}(S)$ is not natural in S, but it is isomorphic to the other group $\text{Aut}(S)$ whenever $S \in SPCat$.

Suppose that $s_1, \ldots$ is a sequence of objects of S satisfying the following cofinality condition: for every $s \in S$ and $n \geq 1$, some $s \square t$ is isomorphic to some $s_n \square \cdots \square s_m$. In this case, we set $\text{Aut}(S) = \text{colim } \text{Aut}(s_1 \square \cdots \square s_n)$.

Since $H_* B(\text{colim } \text{Aut}(s_1 \square \cdots \square s_n)) = \text{colim } H_* B\text{Aut}(s_1 \square \cdots \square s_n)$, the proof of Proposition 1.5 shows that $B_0 S^+$ is the group completion in this context as well.

SCHOLIUM. More is in fact true. If we set $E = [\text{Aut}(S), \text{Aut}(S)]$, then E is a perfect normal subgroup of $\text{Aut}(S)$, and therefore $B_0 S^+$ may also be obtained by applying the "plus construction" to $B\text{Aut}(S)$. It is also the case that $K_2(S) = \pi_2(B_0 S^+)$ equals the Schur multiplier $H_2(E)$. These facts are proven in [We2].

We will conclude this section with a discussion of bisimplicial sets, i.e., functors $X_{*,*} \colon \Delta^{op} \times \Delta^{op} \to Sets$. If we realize vertically to get a simplicial topological space $m \mapsto BX_{m,*}$, and then realize this to get another topological space $|BX_{m,*}|$, we get the realization $|X|$ up to homotopy equivalence (see [Q]). Since we will need them, we insert the following basic results on bisimplicial sets. Typical proofs may be found in [A1], [BF, Appendix B], [BK, Chapter VIII], [M2, Chapter 12], [S], and [Wa, p. 165].

LEMMA 1.7. *Let X_{**}, Y_{**}, and Z_{**} be simplicial sets.*

(i) *If $f \colon X \to Y$ is a map such that $f_{m,*} \colon X_{m,*} \to Y_{m,*}$ is a homotopy equivalence for each $m \geq 0$, then $|f| \colon |X| \to |Y|$ is a homotopy equivalence.*

(ii) *If $X \to Y \to Z$ is a fibration in each degree with connected base $Z_{m,*}$, and $X \to Z$ is constant, then $|X| \to |Y| \to |Z|$ is a fibration.*

(iii) *If each $Z_{m,*}$ is connected, then $|\Omega Z_{m,*}| \simeq \Omega|Z|$.*

COROLLARY 1.8. *If S_* is a simplicial symmetric monoidal category, the realization $|BS_*|$ is an infinite loop space if and only if $\pi_0|BS_*|$ is a group.*

PROOF. There is a simplicial Γ-category C_* with $C_*(1) \simeq S_*$ by the naturality of the construction in Theorem 1.1. Realizing yields a Γ-space $|BC_*|$ with $|BC_*|(1) = |BC_*(1)| \simeq |BS_*|$. The proof ends exactly as the proof of Corollary 1.3 does.

We remark that, since $\pi_0|BS_*|$ is the coequalizer of $\pi_0|BS_1| \rightrightarrows \pi_0|BS_0|$ in the category of monoids, it will be a group if the $\pi_0|BS_i|$ are ($i = 0, 1$).

Here is our application of Corollary 1.8. Let M_* be a simplicial exact category, S_* a simplicial category in *SMCat*. Then BQM_* and BS_*^+ are infinite loop spaces.

2. Karoubi-Villamayor K-theory. In this section we generalize Karoubi and Villamayor's groups $KV_*(R)$, defined in [K-V]. The definitions follow the lead of [A2], as do several of the resulting properties.

If R is a ring with unit, we can define a simplicial ring ΔR as follows: $\Delta_n R$ is the coordinate ring $R[t_0, \ldots, t_n]/(\Sigma t_i = 1)$ of the "standard n-simplex" in $\mathbf{A}_R^{n+1}$. All face and degeneracy maps are determined by their obvious geometric counterparts; for example, $d_i(t_i) = 0$ and $s_i(t_i) = t_i + t_{i+1}$.

We can regard $BGl(\Delta R)$ either as a bisimplicial set or (by vertical realization) as a simplicial topological space which is $BGl(\Delta_n R)$ in degree n. If we apply the plus construction degree-wise, we get a simplicial map $BGl(\Delta R) \to BGl^+(\Delta R)$.

THEOREM 2.1. *The maps $BGl(\Delta R) \to BGl^+(\Delta R) \to B_0\text{Iso}(P(R))^+ \to \Omega_0 BQP(\Delta R)$ of simplicial spaces induce homotopy equivalences on realizations. In particular, for $* \geqslant 1$ we have $KV_*(R) = \pi_*|\Omega_0 BQP(\Delta R)|$.*

The natural map $K_*(R) \to KV_*(R)$ is induced by applying $\Omega_0 BQ\mathbf{P}$ to the diagonal map of simplicial rings $R \to \Delta R$.

PROOF. Anderson has shown in [A2, (1.7)] that $KV_*(R) = \pi_*|BGl(\Delta R)|$ for $* \geqslant 1$, and has observed (in the proof of (2.2)) that $|BGl(\Delta R)|$ is an H-space. Moreover, the natural map $K_*(R) \to KV_*(R)$ is induced by the dotted arrow in the following diagram:

$$
\begin{array}{ccccc}
BGl(R) & \to & BGl^+(R) & \overset{\sim}{\to} & \Omega_0 BQ\mathbf{P}(R) \\
\downarrow & \overset{\cdots}{\epsilon} & \downarrow & & \downarrow \\
BGl(\Delta R) & \overset{\alpha}{\to} & BGl^+(\Delta R) & \overset{\sim}{\to} & \Omega_0 BQ\mathbf{P}(\Delta R).
\end{array}
$$

Since α is acyclic in each degree, the comparison theorem for the Bousfield-Kan spectral sequence (see [BK, p. 339]) shows that the realization of α: $BGl(\Delta R) \to BGl^+(\Delta R)$ is a homology isomorphism. Since it is an H-space map as well, α must be a homotopy equivalence. The result now follows.

REMARK. It follows from Lemma 1.7(iii) that

$$\Omega|BQ\mathbf{P}(\Delta R)| \simeq |\Omega BQ\mathbf{P}(\Delta R)| \simeq |\Omega_0 BQ\mathbf{P}(\Delta R)| \times |K_0(\Delta R)|,$$

so that $BQ\mathbf{P}(\Delta R)$ is not the relevant space to consider, unless perhaps $K_0(R) = K_0(R[t_1, \ldots, t_n])$ for all n.

We are now ready for our generalization. Let ΔR ambiguously denote either the simplicial ring or the category of its morphisms. Note that functors from ΔR correspond to simplicial objects in the receiving category.

DEFINITION 2.2. Let M and S be functors from ΔR to small exact categories and to $SMCat$, respectively. For $* \geqslant 1$ we define the Karoubi-Villamayor K-groups of R with respect to M (or S) to be

$$KV_*M(R) = \pi_*|\Omega_0 BQM(\Delta R)|, \quad KV_*S(R) = \pi_*|B_0 S^+(\Delta R)|.$$

Note that $\Omega_0 BQM(\Delta R)$ and $B_0 S^+(\Delta R)$ are infinite loop spaces by Corollary 1.8.

The definition has of course been rigged so that $KV_*(R) = KV_*\mathbf{P}(R) = KV_*\text{Iso } \mathbf{P}(R)$. If $\mathbf{F}(R)$ denotes the (permutative) subcategory on the modules R^n, i.e., Iso $\mathbf{F}(R) = \amalg Gl_n(R)$, then $KV_*(R) = KV_*\mathbf{F}(R) = KV_*\amalg Gl_n(R)$ as well. This follows from the following result.

THEOREM 2.3 ("COFINALITY"). *Let $M \subset N$ and $S \subset T$ be full cofinal subfunctors from ΔR into small exact categories and into SMCat, respectively. "Cofinal" means that for every $n \in N(\Delta_i R)$ and $t \in T(\Delta_i R)$, some $n \oplus n' \in M(\Delta_i R)$ and some $t \square t' \in S(\Delta_i R)$. Then for all $* \geqslant 1$ the maps $KV_*M(R) \to KV_*N(R)$ and $KV_*S(R) \to KV_*T(R)$ are isomorphisms.*

PROOF. Waldhausen's cofinality theorem on p. 188 of [**Wa**] shows that each $BQM(\Delta_i R) \to BQN(\Delta_i R)$ is a (weak) homotopy equivalence on the basepoint components (the third term in his fibration sequence is a discrete space). Hence $\Omega_0 BQM(\Delta_i R) \to \Omega_0 BQN(\Delta_i R)$ is a homotopy equivalence for each i. Similarly, each $B_0 S^+(\Delta_i R) \to B_0 T^+(\Delta_i R)$ is a homotopy equivalence by Proposition 1.3(ii). The result now follows from Lemma 1.7(i).

PROPOSITION 2.4 ("HOMOTOPY INVARIANCE"). *Let* M, S *be defined on a category* **R** *of rings closed under polynomial adjunction and containing evaluations* $R[x] \to R$ *and* "$x = yz$": $R[x] \to R[y, z]$. *Then for all rings* R *in* **R**, $KV_* M(R) = KV_* M(R[x])$ *and* $KV_* S(R) = KV_* S(R[x])$.

PROOF. The R-algebra maps $h_i: \Delta_n R[x] \to \Delta_{n+1} R[x]$ defined by $h_i(t_j) = s_i(t_j)$, $h_i(x) = x(t_{i+1} + \cdots + t_{n+1})$ provide a homotopy in the category of simplicial rings from the identity map of $\Delta R[x]$ to "$x = 0$": $\Delta R[x] \to \Delta R \to \Delta R[x]$. The notation here is from [**M1**]. The result follows from functoriality and the fact that realization converts simplicial homotopies into topological homotopies.

The relationship between K_* and KV_* is given by the following direct generalization of the Gersten-Anderson spectral sequence for **P**. Since the $B_0 S^+$ and $\Omega_0 BQM$ constructions give connected spaces, this result follows immediately from [**BF**, Appendix B].

THEOREM 2.5. *Let* P *denote a functor from* ΔR *into either exact categories or* SMCat. *Then there is a first quadrant spectral sequence (defined for* $p > 0, q > 1$):

$$E_{pq}^1 = K_q P\big(R[t_1, \ldots, t_p]\big) \Rightarrow KV_{p+q} P(R).$$

In particular, $KV_1 P(R)$ *is the coequalizer of the maps* "$t = 0, 1$": $K_1 P(R[t]) \rightrightarrows K_1 P(R)$.

Since we will need it, we end this section with a discussion of the relative groups $KV_*(R, I)$ for the usual KV theory. If we let $F(R, I)$ denote the basepoint component of the homotopy fiber of the map $BGl^+(R) \to BGl^+(R/I)$, then the homotopy groups $\pi_* F(R, I)$ are the relative terms $K_*(R, I)$ in a long exact sequence for Quillen K-theory. We will let $KV_*(R, I)$ denote the homotopy groups $\pi_* F(\Delta R, \Delta I)$.

THEOREM 2.6 ("EXCISION"). *Let* I *be an ideal of the ring* R. *Then there is a homotopy equivalence* $|BGl(\Delta I)| \simeq |F(\Delta R, \Delta I)|$. *In particular, the groups* $KV_*(R, I)$ *are independent of the ambient ring* R, *and may be computed simplicially as* $KV_*(R, I) = \pi_{*-1} Gl(\Delta I)$. *Moreover, there is a first quadrant spectral sequence* ($p \geqslant 0, q \geqslant 1$):

$$E_{pq}^1 = K_p\big(R[t_1, \ldots, t_q], I[t_1, \ldots, t_q]\big) \Rightarrow KV_{p+q}(I).$$

REMARK. This theorem implies that the above definition of $KV_*(I)$ agrees with the one given in [**K-V**]. We will see in the next section how to fit the groups $KV_*(I)$ into a long exact sequence when $R \to R/I$ is not a Gl-fibration.

PROOF. Define the simplicial group A_* by exactness of the sequence

$$1 \to Gl(\Delta I) \to Gl(\Delta R) \to Gl(\Delta R/I) \to A_* \to 1.$$

Then $A_* \times BGl(\Delta I) \to BGl(\Delta R) \to BGl(\Delta R/I)$ is a fibration of simplicial spaces, since it is one on each level (check the long exact homotopy sequences). On the other hand, there is a fibration (again by Lemma 1.7)

$$A_* \times F(\Delta R, \Delta I) \to BGl^+(\Delta R) \to BGl^+(\Delta R/I).$$

Now there is an evident map between these fibrations, which is a homotopy equivalence on the base and total space by Theorem 2.1. It must then be a homotopy equivalence on the fibers. Splitting off the common factor of $|A_*|$ gives the homotopy equivalence we want. Finally, the spectral sequence for $KV_*(R, I)$ is also an application of the spectral sequence given in Appendix B of [BF].

3. Exact categories. Much of the power of Quillen's Q-construction is a result of the ability to pass between exact categories, comparing their K-theories. In this section, we show that the Cartan map $K_* \to G_*$ factors through KV_*. We also show how the localization sequences of K-theory carry over to the KV-theory.

If R is a noetherian ring, the finitely generated R-modules form an abelian category $\mathbf{M}(R)$, whose K-theory is denoted $G_*(R)$. This is not entirely appropriate for our setting, as there are no exact maps $\mathbf{M}(R[t]) \to \mathbf{M}(R)$ splitting the inclusion. In order to get a functor $\Delta R \to$ (small exact categories), we need a technical dodge.

Define $\mathbf{M}_p$ to be the full exact subcategory of $\mathbf{M}(\Delta_p R)$ of all modules M satisfying $\mathrm{Tor}_*(\Delta_q R, M) = 0$ for $* \neq 0$ and for all ring maps $\Delta_p R \to \Delta_q R$ in the category ΔR. Note that only the ring maps with $q < p$ need be checked, and that $\mathbf{M}_0 = \mathbf{M}(R)$. If $M \in \mathbf{M}_p$ and $\Delta_p R \to \Delta_q R$ is in ΔR, then $\Delta_q R \otimes M$ belongs to $\mathbf{M}_q$, since the spectral sequence for $\Delta_p R \to \Delta_q R \to \Delta_r R$ degenerates to give (for $* \neq 0$)

$$0 = \mathrm{Tor}_*^{\Delta_p R}(\Delta_r R, M) = \mathrm{Tor}_*^{\Delta_q R}(\Delta_r R, \Delta_q R \otimes M).$$

Thus $\Delta_q R \otimes$ is an exact functor from $\mathbf{M}_p$ to $\mathbf{M}_q$, and $\Delta_p R \mapsto \mathbf{M}_p$ gives a functor from ΔR to exact categories, which we will denote $\mathbf{M}_*$. A useful observation is that the constant simplicial exact category $\mathrm{Diag}(\mathbf{M}(R))$ maps to $\mathbf{M}_*$, the degree p part being $\Delta_p R \otimes : \mathbf{M}(R) \to \mathbf{M}_p$.

Every module in $\mathbf{M}(\Delta_p R)$ has a finite resolution by modules in $\mathbf{M}_p$ (the pth Syzygy of a free resolution lies in $\mathbf{M}_p$), so by Quillen's resolution theorem $BQ\mathbf{M}_p \simeq BQ\mathbf{M}(\Delta_p R)$. Thus $\mathbf{M}_*$ is an adequate substitute for the fictitious $\mathbf{M}(\Delta R)$. Moreover, $BQ\mathbf{M}(R) \to BQ\mathbf{M}(\Delta_p R)$ is a homotopy equivalence (see [Q, p. 122]), so the map from the constant simplicial space $\mathrm{Diag}(BQ\mathbf{M}(R))$ to $BQ\mathbf{M}_*$ is a homotopy equivalence in each degree. By Lemma 1.7 we have $\Omega_0 BQ\mathbf{M}(R) \simeq \Omega_0 BQ\mathbf{M}_*$.

Now the inclusion $\mathbf{P} \subset \mathbf{M}$ extends to $\mathbf{P}(\Delta R) \subset \mathbf{M}_*(\Delta R)$, and we have a diagram of simplicial spaces:

$$
\begin{array}{ccc}
\Omega_0 BQ\,\mathbf{P}(R) & \to & \Omega_0 BQ\,\mathbf{M}(R) \\
\downarrow & & \downarrow \sim \\
\Omega_0 BQ\,\mathbf{P}(\Delta R) & \to & \Omega_0 BQ\,\mathbf{M}_*(\Delta R)
\end{array}
$$

By the above remarks, the right vertical arrow is a homotopy equivalence. Theorem 2.1 gives the following result:

THEOREM 3.1. *The Cartan map $K_*(R) \to G_*(R)$ factors through the natural map $K_*(R) \to KV_*(R)$ for every noetherian ring R.*

In the remainder of his section we show that, if suitable care is taken, we can extend the localization sequences to the Karoubi-Villamayor theory. (Of course other tools work as well, such as characteristic exact filtration, resolution, and devissage. This is immediate from the comparison theorem for the spectral sequence (2.5).)

Recall that for a group-valued functor F defined on rings, $NF(R)$ denotes the kernel of "$t = 0$": $F(R[t]) \to F(R)$. By iteration, $N^n F(R)$ is the subgroup of all $g(t_1, \dots, t_n) \in F(R[t_1, \dots, t_n])$ with $g(0, t_2, \dots, t_n) = \cdots = g(t_1, \dots, t_{n-1}, 0) = 1$. By convention, $N^0 F = F$.

Given a functor F from ΔR to (groups), we can still make sense of the definition of $N^n F(R)$ as a subgroup of $F(\Delta_n R)$. The map $F(d_n)$ induces a map $N^n F(R) \to N^{n-1} F(R)$, and $N^* F(R)$ is a (not necessarily abelian) chain complex whose homology is the homotopy of the simplicial group $F(\Delta R)$. A proof may be found on p. 69 of [M1]; $N^* F(R)$ is called the Moore complex associated to $F(\Delta R)$.

THEOREM 3.2. *Let I be an ideal of the ring R, and define the chain complex C_* by exactness of $N^n K_1(R) \to N^n K_1(R/I) \to C_n \to 0$. Then there is a long exact sequence*

$$\dots KV_{*+1}(R/I) \to H_*(C) \oplus KV_*(I) \to KV_*(R) \to \dots$$

ending in

$$KV_1(R) \to KV_1(R/I) \to K_0(I)/\mathrm{im}(NK_1(R/I)) \to K_0(R) \to K_0(R/I).$$

PROOF. From the proof of Theorem 2.6 there is a fibration

$$A \times BGl(\Delta I) \to BGl(\Delta R) \to BGl(\Delta R/I).$$

The Moore complex associated to A is C, while $\pi_* BGl(\Delta R) = KV_*(R)$ and $\pi_* BGl(\Delta I) = KV_*(I)$ for $* \neq 0$ by Theorems 2.1 and 2.6. The sequence of the theorem is thus the homotopy sequence of the fibration, except for the ending:

$$KV_1(R) \to KV_1(R/I) \to C_0/\mathrm{im}(NK_1(R/I) \to C_0) \to 0.$$

Since C_0 is a subgroup of $K_0(I)$, the ending in the theorem follows from the usual ideal sequence for K_1, K_0.

REMARK. In [K-V], Karoubi and Villamayor call $R \to R/I$ a "Gl-fibration" just in case $C_* = 0$ for $* \neq 0$, and gave the sequence in this instance.

PROPOSITION 3.3. *Let $M_i \colon \Delta R \to$ (exact categories) be such that $BQM_0 \to BQM_1 \to BQM_2$ is a fibration at each ring of ΔR. Then there is a long exact sequence*

$$\dots KV_{*+1}M_2(R) \to H_*(C) \oplus KV_*M_0(R) \to KV_*M_1(R) \dots$$

ending in

$$KV_1 M_1 \to KV_1 M_2 \to K_0 M_0/\mathrm{im}\, C_1 \to K_0 M_1 \to K_0 M_2.$$

Here C_* is the chain complex defined by exactness of $N^n K_1 M_1 \to N^n K_1 M_2 \to C_n \to 0$.

Note that C_n is also the kernel of $N^n K_0 M_0(R) \to N^n K_0 M_1(R)$. In particular, if either $K_1 M_2$ or $K_0 M_0$ is constant on ΔR then $C_n = 0$ for $n \neq 0$ and we obtain the "expected" long exact sequence of KV-groups.

PROOF. Let A_* be the cokernel of $K_1 M_1(\Delta R) \to K_1 M_2(\Delta R)$, i.e., the kernel of $K_0 M_0(\Delta R) \to K_0 M_1(\Delta R)$. Then the sequence of simplicial spaces $A \times \Omega_0 BQM_0 \to \Omega_0 BQM_1 \to \Omega_0 BQM_2$ is a fibration at each level, hence on total spaces by Lemma 1.7(ii). The long exact homotopy sequence is

$$\ldots \pi_{*+1} \Omega_0 BQM_2 \to \pi_* A \oplus \pi_* \Omega_0 BQM_0 \to \pi_* \Omega_0 BQM_1 \ldots$$

ending in $\pi_1 \Omega_0 BQM_2 \to \pi_0 A \to 0$.

Since C_* is the Moore complex associated to A_*, $H_* C = \pi_* A$, and we have proven all but exactness of the ending, for which we splice the exact sequences $K_1 M_1 \to K_1 M_2 \to C_0 / \mathrm{im}\, C_1 \to 0$ and $0 \to C_0 \to K_0 M_0 \to K_0 M_1 \to K_0 M_2$.

COROLLARY 3.4. *Let S be a multiplicative set of central nonzero divisors of the ring R, and suppose $K_1(R_S) = K_1(R_S[t_1, \ldots, t_n])$ for all n. Then there is an exact sequence $\ldots KV_* \mathbf{H}_S(R) \to KV_*(R) \to KV_*(R_S) \ldots$, where $\mathbf{H}_S(R)$ is the category of finitely generated R-modules M of finite projective dimension satisfying $M_S = 0$.*

REMARK. This localization sequence is utilized in [We3] to compute K_2 of a seminormal curve over an algebraically closed field.

COROLLARY 3.5 (LOCALIZATION). *Let $\mathbf{A}: \Delta R \to$ (abelian categories) have a Serre subcategory $\mathbf{B}$. Then there is a long exact sequence*

$$\ldots KV_{*+1} \mathbf{A}/\mathbf{B}(R) \to H_*(C) \oplus KV_* \mathbf{B}(R) \to KV_* \mathbf{A}(R) \ldots,$$

and the $H_*(C)$ terms vanish if the $N^n K_1 \mathbf{A} \to N^n K_1 \mathbf{A}/\mathbf{B}$ are onto for $n \neq 0$, e.g., if $K_1 \mathbf{A}/\mathbf{B}(R) = K_1 \mathbf{A}/\mathbf{B}(R[t_1, \ldots, t_n])$ for all n.

4. Monoidal categories. In this section we develop a simplicial technique for computing the groups $KV_* S(R)$, directly analogous to Karoubi and Villamayor's original construction for $KV_*(R)$, as modified by Rector (see [A2]). The idea is that the simplicial groups $\mathrm{Aut}(S(\Delta R))$ are often deloopings of the infinite loop spaces $B_0 S^+(\Delta R)$. One can then use simplicial techniques to compute these groups.

LEMMA 4.1. *If $S \in SPCat$, there is a group homomorphism $\square: \mathrm{Aut}(S) \times \mathrm{Aut}(S) \to \mathrm{Aut}(S)$.*

PROOF. This is a generalization of the corresponding result for Gl (q.v. [Wg]). Let T denote the translation category of $\pi_0 S$. The hypothesis that $S \in SPCat$ ensures that there are functors $s \mapsto \mathrm{Aut}(s) \times \mathrm{Aut}(s)$ and $s \mapsto \mathrm{Aut}(s \square s)$ from T to groups, and that $\square: S \times S \to S$ induces a natural transformation between them. The directed colimits of the two functors are $\mathrm{Aut}(S) \times \mathrm{Aut}(S)$ and $\mathrm{Aut}(S)$, respectively, and the natural transformation induces the desired group homomorphism.

In general, if $S: \Delta R \to SPCat$, we cannot compute the groups $KV_* S(R)$ directly from $\mathrm{Aut}(S(\Delta R))$. The problem is that $\pi_0 \mathrm{Aut}(S(\Delta R))$ need not be abelian, as the case S constant shows. This is the only difficulty that can arise.

We will say that "the Whitehead lemma" holds for $S(R)$ if for each $\alpha \in \mathrm{Aut}(s)$, $s \in S(R)$, there is some $\beta(t) \in \mathrm{Aut}(S(R[t]))$ with $\beta(0) = 1$ and $\beta(1) = \alpha \square \alpha^{-1}$.

THEOREM 4.2. *The following are equivalent for* S: $\Delta R \to SPCat$:
(i) *The Whitehead lemma holds for* $S(R)$.
(ii) $\pi_0 \mathrm{Aut}(S(\Delta R))$ *is abelian.*
(iii) $|\mathrm{Aut}(S(\Delta R))| \simeq \Omega B_0 S^+(\Delta R)$ *(an infinite loop space).*
In this case, $KV_* S(R) = \pi_{*-1} \mathrm{Aut}(S(\Delta R))$ *for all* $* \geqslant 1$. *In particular,* $KV_1 S(R)$ *is the quotient of* $\mathrm{Aut}(S(R))$ *by the equivalence relation* $\beta(0) \sim \beta(1)$ *for all* $\beta(t_1) \in$ $\mathrm{Aut}(S(R[t_1]))$.

PROOF. Since $\alpha \square \alpha^{-1}$ is a commutator (see [**B2**, p. 351]) and $\alpha\beta\square1 = (\alpha\square\alpha^{-1})(\beta\alpha\square1)(\alpha^{-1}\square\alpha)$, it is clear that (i) and (ii) are equivalent and are implied by (iii).

Conversely, we claim that (ii) implies that $B\mathrm{Aut}\,S(\Delta R)$ is an H-space. Assuming this for the moment, there is a map $B_0 S^+ \to B\mathrm{Aut}\,S(\Delta R)$ by Proposition 1.5. The argument of Theorem 2.1 now applies to show that $B\mathrm{Aut}\,S(\Delta R) \simeq B_0 S^+(\Delta R)$. Since $\pi_{*-1}\mathrm{Aut}\,S(\Delta R) = \pi_* B\mathrm{Aut}\,S(\Delta R)$, e.g., by Lemma 1.7(ii), the last few assertions of the theorem are seen by unravelling the appropriate definitions.

We will prove that $B\mathrm{Aut}\,S(\Delta R)$ is an H-space by adapting Wagoner's proof in [**Wg**] that $(B\mathrm{Aut}\,S)^+$ is an H-space. For convenience, set $G = \mathrm{Aut}\,S(\Delta R)$, and let E be the simplicial subgroup of G which maps to $1 \in \pi_0|G|$. Then BE is a covering space of BG with fiber $\pi_1 BG$, i.e., the universal cover. $\mathrm{Aut}\,S(R)$, and hence $\pi_1 BG$, acts trivially on the homology of BE since conjugation by $\alpha \in \mathrm{Aut}\,S(R)$ agrees with conjugation by $\alpha\square\alpha^{-1}$ (which is in E_0 by (ii)) on any finite subset of E. This shows that BG is weakly simple. Now consider $\phi\colon G \to G$ given by $\phi(g) = g\square1$. Again, ϕ acts by conjugation on any finite subset of G, so ϕ_* is the identity map on $H_* BG$. Thus $B\phi$ is a homotopy equivalence (see Lemma 1.1 of [**Wg**]). Similarly, $B\psi$ is an h.e., where $\psi(g) = 1\square g$. Now consider the map

$$m = (B\square)(B\phi^{-1} \times B\psi^{-1})\colon BG \times BG \to BG,$$

where $\square$ is the map of Lemma 4.1. When restricted to the left and right factors, m is homotopic to the identity, so m is an H-space multiplication on BG.

EXAMPLE 4.3. For any commutative ring R, let $\mathrm{Pic}(R)$ denote the permutative category of rank one projective R-modules under $\otimes$. For each L, $\mathrm{Aut}(L) = U(R)$ $=$ (units of R) and translation is constant. It follows that $\mathrm{Aut}(\mathrm{Pic}(R)) = U(R)$ and that $B_0\mathrm{Pic}(R)^+ = BU(R)$. Hence the groups $K_*\mathrm{Pic}(R)$ are zero for $* \geqslant 2$, and $K_1\mathrm{Pic}(R) = U(R)$.

Let $[U]R$ denote $U(R)/(1 + \mathrm{nil}(R))$, and note that $[U]R = [U]R[t]$. The sequence $1 + \mathrm{nil}(\Delta R) \to U(\Delta R) \to [U]R$ is a fibration, and $1 + \mathrm{nil}(\Delta R)$ is contractible ($1 + t_{n+1}f$ is a homotopy from 1 to $1 + f$). Hence $KV_*\mathrm{Pic}(R) = 0$ for $* \geqslant 2$, and $KV_1\mathrm{Pic}(R) = [U]R$.

EXAMPLE 4.4. Let R be a ring with involution, λ a central element of R satisfying $\lambda\bar\lambda = 1$, and Λ an additive subgroup of $\{a \in R\colon a = -\lambda\bar a\}$ containing $\{a - \lambda\bar a\}$ and closed under $r \to ar\bar a$. Then the category $\mathbf{Quad}^\lambda(R, \Lambda)$ of nonsingular (λ, Λ)-quadratic R-modules was defined in [**B3**]. It is a member of $SMCat$, and its product is direct sum. By turning to the hyperbolic modules we can construct a member of $SPCat$, which allows us to take $\mathrm{Aut}(\mathbf{Quad}^\lambda(R, \Lambda))$ to be the unitary group $U^\lambda(R, \Lambda)$. The resulting group $K_*\mathbf{Quad}^\lambda(R)$ are the homotopy groups of

$BU^\lambda(R)^+$ and are denoted $_\lambda L_*(R)$ in [K]. The Karoubi-Villamayor groups $KV_*\mathbf{Quad}^\lambda(R)$ are the simplicial homotopy groups of $U^\lambda(\Delta R)$ by Theorem 4.2, and agree with the groups $_\lambda L^{-*}(R)$ defined by Karoubi in [K].

5. Computations. In this section we generalize the results of [B1], computing $KV_*S(R)$ for $S = \mathbf{FP}$ and $\mathbf{Az}$. We use the simplicial methods of §4 in comparing these groups to the "standard": $KV_*(R)$.

It is perhaps apropos to explain the underlying philosophy at the onset. It is the case that both the Quillen groups $K_*\mathbf{FP}$ and $K_*\mathbf{Az}$ are known–the former was computed by J. P. May in [M4], and the latter is computed in [We2]. Using the spectral sequence of Theorem 2.5 and a bit of care, we could compute $KV_*\mathbf{FP}$ and $KV_*\mathbf{Az}$, and that would be that. However, such a proof uses a lot of heavy machinery that can be avoided. It is the purpose of this section to provide an elementary algebraic computation that refers to nothing more difficult than the long exact homotopy sequence of a simplicial fibration (q.v. [M1, p. 27]).

In fact, the computation of $KV_*\mathbf{Az}$ in this section preceded and motivated the computation of $K_*\mathbf{Az}$ in [We2]. This illustrates the philosophy that the KV-computations should anticipate and guide the K-computations.

We now turn to the category $\mathbf{FP}(R)$ of finitely generated faithful projective R-modules under $\otimes$. $\mathbf{FP}$ is a functor from commutative rings into $SMCat$.

The full cofinal subcategory $\mathbf{FF}(R)$ of $\mathbf{FP}(R)$ on the objects R^n, $n \geqslant 1$, can be given the following product structure. The product $R^m \otimes R^n$ is identified with R^{mn} by identifying the basis elements $e_i \otimes e_j$ and e_k, where $k = (i - 1)n + j$. Although not a permutative category, it is a straightforward matter to check that $\mathbf{FF}(R)$ belongs to $SPCat$, and that $\mathrm{Aut}(\mathbf{FF}(R))$ is the group $\mathrm{Gl}_\otimes(R)$.

THEOREM 5.1. *For $* \geqslant 1$, $KV_*\mathbf{FP}(R) = \mathbf{Q} \otimes KV_*(R)$.*

We first note that the Whitehead lemma holds for $\mathbf{FF}(R)$. Indeed, if τ_1 is the transposition of basis elements of $\mathbf{R}^n \otimes \mathbf{R}^n$ for which $\tau_1(\alpha \otimes 1)\tau_1^{-1} = 1 \otimes \alpha$, then τ_1 is elementary whenever n is divisible by 4 (see [B1]). We can find $\tau(t)$ in $\mathrm{Gl}_{n^2}(R[t])$ with $\tau(0) = 1$ and $\tau(1) = \tau_1$. The desired contraction of $\alpha \otimes \alpha^{-1}$ is given by the element $\tau(\alpha \otimes 1)\tau^{-1}(1 \otimes \alpha^{-1})$ of $\mathrm{Gl}_\otimes(R[t])$.

Now define maps f_{mn}, g_{mn} from $\mathrm{Gl}_m(\Delta R)$ to $\mathrm{Gl}_{mn}(\Delta R)$ by the formulas $f_{mn}(\alpha) = \mathrm{diag}(\alpha, I, \ldots, I)$, $g_{mn}(\alpha) = \mathrm{diag}(\alpha, \ldots, \alpha)$. By the Whitehead lemma, the maps $n \cdot f_{mn}$ and g_{mn} are homotopic when $n \geqslant 4$. Copying the argument of [B1, p. 42], we get

$$\pi_*\mathrm{Gl}_\otimes(\Delta R) = \mathrm{colim}\, \pi_*(g_{mn}) = \mathbf{Q} \otimes \mathrm{colim}\, \pi_*(f_{mn}) = \mathbf{Q} \otimes \pi_*\mathrm{Gl}(\Delta R).$$

Theorem 5.1 is immediate from this, given Theorem 4.2.

REMARK 5.2. In fact, it is implicit in [M4, p. 96] that for $* \geqslant 1$, $K_*\mathbf{FP}(R) = \mathbf{Q} \otimes K_*(R)$: Since $\mathbf{FF}$ forms a "bipermutative category" when combined with $\mathbf{F}$, May's result states that $B\mathbf{F}^+ \to B\mathbf{FF}^+$ is a localization, which specializes on homotopy to the $K_*\mathbf{FP}$ result.

We conclude this section by computing the KV-theory of the category $\mathbf{Az}(R)$ of Azumaya algebras over the commutative ring R. The idea here is to show that the natural transformation $\mathrm{End}: \mathbf{FP} \to \mathbf{Az}$ often induces an isomorphism of KV-groups.

Let $\mathbf{MAz}(R)$ be the full subcategory of $\mathbf{Az}(R)$ on the matrix rings $M_m(R)$. This is cofinal, and can be endowed with a product in such a way that End: $\mathbf{FF} \to \mathbf{MAz}$ is a strict morphism in *SPCat*. In this way we obtain a transformation End: $\mathrm{Gl}_\otimes = \mathrm{Aut}(\mathbf{FF}) \to \mathrm{Aut}(\mathbf{MAz}) = \mathrm{Aut}(\mathbf{Az})$ of group-valued functors. Recall from [**B1**], [**B2**, p. 74]:

PROPOSITION 5.3 (ROSENBERG-ZELINSKY). *There is an exact sequence* $1 \to U(R) \to \mathrm{Gl}_\otimes(R) \to \mathrm{Aut}(\mathbf{Az}(R)) \to \mathrm{TPic}(R) \to 1$, *where* $\mathrm{TPic}(R)$ *denotes the torsion subgroup of* $\mathrm{Pic}(R)$.

Now recall that a map of simplicial groups $H_* \to G_*$ is a fibration if and only if the cosets $(G/H)_*$ constitute a constant simplicial set. In the case at hand this translates to

COROLLARY 5.4. End: $\mathrm{Gl}_\otimes(\Delta R) \to \mathrm{Aut}(\mathbf{Az}(\Delta R))$ *is a fibration iff* $\mathrm{TPic}(R) = \mathrm{TPic}(R[t_1, \ldots, t_n])$ *for all n. The fiber of* End *is* $U(\Delta R)$.

We can now state the generalization of Bass' computation that

$$0 \to (\mathbf{Q}/\mathbf{Z} \otimes U(R)) \oplus (\mathbf{Q} \otimes SK_1(R)) \to K_1\mathbf{Az}(R) \to \mathrm{TPic}(R) \to 0$$

is split exact. Our notation is that $KV_1 = [U] \oplus SKV_1$, and that $T[U]$ denotes the torsion subgroup of $[U]$.

THEOREM 5.5. *Suppose* $\mathrm{TPic}(R) = \mathrm{TPic}(R[t_1, \ldots, t_n])$ *for all n. Then*

$$KV_*\mathbf{Az}(R) = KV_*\mathbf{FP}(R) = \mathbf{Q} \otimes KV_*(R) \quad \text{for } * \geqslant 3,$$

and the following sequences are split exact:

$$0 \to \mathbf{Q} \otimes KV_2(R) \to KV_2\mathbf{Az}(R) \to T[U]R \to 0,$$

$$0 \to (\mathbf{Q}/\mathbf{Z} \otimes [U]R) \oplus (\mathbf{Q} \otimes SKV_1(R)) \to KV_1\mathbf{Az}(R) \to \mathrm{TPic}(R) \to 0.$$

PROOF. We first observe that the Whitehead lemma holds for $\mathrm{Aut}(\mathbf{Az}(R))$. This may be seen by applying End to the argument for

$$\mathrm{Gl}_\otimes: \mathrm{End}(\tau)(\alpha \otimes 1)\mathrm{End}(\tau^{-1})(1 \otimes \alpha)$$

is a contraction of the R-algebra automorphism $\alpha \otimes \alpha^{-1}$. Thus $KV_*\mathbf{Az}(R) = \pi_{*-1}\mathrm{Aut}(\mathbf{Az}(\Delta R))$. The long exact homotopy sequence for the fibration End is

$$\ldots \pi_* U(\Delta R) \to \pi_*\mathrm{Gl}_\otimes(\Delta R) \to \pi_*\mathrm{Aut}(\mathbf{Az}(\Delta R)) \ldots.$$

Making use of the computation of $\pi_* U(\Delta R)$ in Example 4.3, we find that $KV_*\mathbf{Az} = KV_*\mathbf{FP}$ for $* \geqslant 3$, and that there is an exact sequence

$$0 \to KV_2\mathbf{FP}(R) \to KV_2\mathbf{Az}(R) \to [U]R \to KV_1\mathbf{FP}(R) \to KV_1\mathbf{Az}(R).$$

From (5.3) and the description $KV_1\mathbf{Az}(R) = \pi_0\mathrm{Aut}\, \mathbf{Az}(\Delta R)$, we find that the cokernel of $KV_1\mathbf{FP} \to KV_1\mathbf{Az}$ is TPic. The claimed sequences follow from Theorem 5.1, and split because the left group is divisible in each case.

REMARK 5.6. Using the computation of $KV_*\mathbf{Az}$ as motivation, the author has recently shown that $K_*\mathbf{Az} = K_*\mathbf{FP}$ for $* \geqslant 3$, and that $K_2\mathbf{Az} = K_2\mathbf{FP} \oplus$ (roots of unity). The proof uses entirely different techniques and appears in [**We2**].

REMARK 5.7. The image $[TU]R$ of the roots of unity in $T[U]R$ may not be the entire group, since a unit u of R may have some u^n unipotent without u itself being unipotent. An example is provided by u in the ring $R = \mathbf{Z}[u]$ with $(u^n - 1)^2 = 0$.

This phenomenon provides an example in which the spectral sequence of Theorem 2.5 does not completely degenerate. By Remark 5.6, $[TU]R$ is a summand of $E_{02}^2(\mathbf{Az})$, while $T[U]R/[TU]R$ is a summand of E_{11}^2. In the present example, the class of u in E_{11}^2 is represented by the element $n^{-1} \otimes (1 + t(u^n - 1))$ of the summand $\mathbf{Q}/\mathbf{Z} \otimes U(R[t])$ of $K_1\mathbf{Az}(R[t])$.

When $\mathrm{TPic}(R) \neq \mathrm{TPic}(R[t_1, \ldots, t_n])$ the following changes need to be made. There are two fibrations, End: $\mathrm{Gl}_\otimes(\Delta R) \to \mathrm{PGl}_\otimes(\Delta R)$ and $\mathrm{PGl}_\otimes(\Delta R) \to \mathrm{Aut}(\mathbf{Az}(\Delta R)) \to \mathrm{TPic}(\Delta R)$, and the Whitehead Lemma holds for $\mathrm{PGl}_\otimes(\Delta R)$. In fact there is a subcategory $\mathbf{In}$ of $\mathbf{Az}$ consisting of inner automorphisms (q.v. [**We2**]), and $\mathrm{Aut}(\mathbf{In}) = \mathrm{PGl}_\otimes$. The long exact homotopy sequence of the first fibration yields

$$KV_*\mathbf{In}(R) = \mathbf{Q} \otimes KV_*(R), \quad * \geq 3,$$

$$KV_2\mathbf{In}(R) = \mathbf{Q} \otimes KV_2(R) \oplus T[U]R,$$

$$KV_1\mathbf{In}(R) = (\mathbf{Q}/\mathbf{Z} \otimes [U]R) \oplus (\mathbf{Q} \otimes SKV_1(R)).$$

The homotopy sequence of the second fibration reads

$$\ldots KV_*\mathbf{In}(R) \to KV_*\mathbf{Az}(R) \to \pi_{*-1}\mathrm{TPic}(\Delta R) \ldots$$

and ends with $KV_1\mathbf{Az}(R) \to [\mathrm{TPic}]R \to 0$, where $[\mathrm{TPic}]R$ is the coequalizer of the maps "$t = 0, 1$": $\mathrm{TPic}(R[t]) \rightrightarrows \mathrm{TPic}(R)$. Because the $\pi_*\mathrm{TPic}(\Delta R)$ are torsion groups and (for $* \geq 3$) the $KV_*\mathbf{In}$ are torsionfree and divisible, the long exact sequence splits up into short split exact sequences. We know $KV_1\mathbf{Az}$ from Bass' computation of $K_1\mathbf{Az}$. We obtain

THEOREM 5.8. *For* $* \geq 3$, $KV_*\mathbf{Az}(R) = \mathbf{Q} \otimes KV_*(R) \oplus \pi_{*-1}\mathrm{TPic}(\Delta R)$. *We have* $KV_1\mathbf{Az} = KV_1\mathbf{In}(R) \oplus [\mathrm{TPic}]R$, *and* $KV_2\mathbf{Az}(R) = \mathbf{Q} \otimes KV_2(R) \oplus KV_2^{\mathrm{tor}}(R)$. *The torsion subgroup* $KV_2^{\mathrm{tor}}(R)$ *of* $KV_2\mathbf{Az}(R)$ *is determined up to extension by the exact sequence*

$$0 \to T[U]R \to KV_2^{\mathrm{tor}}(R) \to \pi_1\mathrm{TPic}(\Delta R) \to 0.$$

I do not know if the sequence for $KV_2^{\mathrm{tor}}(R)$ splits.

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